APPLICATION OF COMPARISON THEOREMS TO THE THEORY OF HEAT CONDUCTION

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Upper and lower estimates are established for the solutions to certain boundary-value problems in the theory of heat conduction.

As is well known, comparison theorems yield "from below" and "from above" estimates in the solution of differential equations. With sufficiently accurate estimates based on comparison theorems, it is possible to evaluate certain approximate methods of solution.

In the technical literature are given comparison theorems for ordinary differential equations, whether linear [1] or nonlinear [2]. In the case of self-adjoint and limit self-adjoint solutions to parabolic equations, one can establish estimates for the solution on the basis of comparison theorems for ordinary differential equations. The comparison theorem proved in [3, 4] applies to parabolic equations with any initial and boundary conditions. The comparison theorem proved in [5] applies to a specific system of the parabolic kind. The use of comparison theorems for estimating the solutions to self-adjoint problems has been dealt with in [6, 7]. In [5-6] comparison theorems are used for estimating the solutions as well as for analyzing the accuracy of the linearization method and of the method of successive steady states.

Here the authors will show on the specific examples how the Westphal theorem can be applied to the theory of heat conduction.

1. We will estimate the solution to the problem of heat propagation through a semiinfinitely large medium with a temperature-dependent thermal conductivity varying along the space coordinate:

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(a^2 T x^\alpha \frac{\partial T}{\partial x}\right), \tag{1}$$

$$T(0,\ t) = T_c;\quad T(\infty,\ t) = T_0;\quad T(x,\ 0) = T_0. \tag{2}$$

We introduce the function $u = T^2$ which represents the solution to the following problem:

$$\frac{\partial u}{\partial t} = a^2 \sqrt{u}\frac{\partial}{\partial x}\left(x^\alpha \frac{\partial u}{\partial x}\right), \tag{3}$$

$$u(0,\ t) = T_c^2 = u_c;\quad u(\infty,\ t) = T_0^2 = u_0;\quad u(x,\ 0) = u_0. \tag{4}$$

We assume that $\partial T/\partial t \le 0$ (when $T_c < T_0$), then $\partial u/\partial t \le 0$.

We next consider the functions u_1 and u_2 which satisfy respectively the following equations:

$$\frac{\partial u_1}{\partial t} = a^2 \sqrt{u_c}\frac{\partial}{\partial x}\left(x^\alpha \frac{\partial u_1}{\partial x}\right),\quad \frac{\partial u_2}{\partial t} = a^2 \sqrt{u_0}\frac{\partial}{\partial x}\left(x^\alpha \frac{\partial u_2}{\partial x}\right) \tag{5}$$

and conditions (4). Obviously, the following inequalities

$$\begin{aligned} \frac{\partial u_1}{\partial t} &\geqslant a^2 \sqrt{u_1}\frac{\partial}{\partial x}\left(x^\alpha \frac{\partial u_1}{\partial x}\right), \\ \frac{\partial u_2}{\partial t} &\leqslant a^2 \sqrt{u_2}\frac{\partial}{\partial x}\left(x^\alpha \frac{\partial u_2}{\partial x}\right), \end{aligned} \tag{6}$$

hold true.

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As a consequence of the Westphal theorem, it follows from (6) that

$$u_2 \leqslant u \leqslant u_1. \tag{7}$$

The solutions to Eq. (5) under conditions (4) are

$$u_i(x,\ t) = A_i + B_i \int\limits_0^{x^{\alpha-2}t} y^{-\frac{2\alpha-3}{\alpha-2}} \exp\left(-\frac{1}{b_i(\alpha-2)^2 y}\right) dy,$$

where

$$i = 1,\ 2;\quad A_1 = A_2 = u_0;\quad B_i = \frac{u_c - u_0}{\int\limits_0^{\infty} y^{-\frac{2\alpha-3}{\alpha-2}} \exp\left(-\frac{1}{b_i(\alpha-2)^2 y}\right) dy};$$

$$b_1 = a^2\sqrt{u_c};\quad b_2 = a^2\sqrt{u_0}.$$

By virtue of (7), for the thermal fluxes at x = 0 we have the estimate

$$q_2 \leqslant q \leqslant q_1.$$

The ratio $\gamma = q_1/q_2$ was calculated for $\alpha = 1/2$.

For $\alpha = 1.2$, 1.5, and 2.0 we have obtained $\gamma = 1.07$, 1.18, and 1.29 respectively.

2. We will next consider the problem of heat propagation in a two-layer plate. The heat transfer between layers is assumed to obey Newton's Law. The heat propagation process is described by the following system of equations:

$$\frac{\partial T_1}{\partial t} = a_1^2 \frac{\partial^2 T_1}{\partial x^2} - \alpha_1 (T_1 - T_2),$$
$$\frac{\partial T_2}{\partial t} = a_2^2 \frac{\partial^2 T_2}{\partial x^2} - \alpha_2 (T_2 - T_1), \tag{8}$$

where T_1 and T_2 are the temperatures of the respective layers. The initial and the boundary conditions are

$$T_1(0,\ t) = 0;\quad T_1(x,\ 0) = 0;\quad T_1(1;\ t) = 1; \tag{9}$$
$$T_2(0,\ t) = 0;\quad T_2(x,\ 0) = 0;\quad T_2(1,\ t) = 1. \tag{10}$$

We assume the following inequalities:

$$0 \leqslant T_1 \leqslant 1;\quad 0 \leqslant T_2 \leqslant 1.$$

Let functions T_{11} and T_{12} represent respectively the solutions to the equations

$$\frac{\partial T_{11}}{\partial t} = a_1^2 \frac{\partial^2 T_{11}}{\partial x^2} - \alpha_1,$$
$$\frac{\partial T_{12}}{\partial t} = a_2^2 \frac{\partial^2 T_{12}}{\partial x^2} \tag{11}$$

and satisfy conditions (9). For functions T_{11} and T_{12} one can write inequalities of the (6) kind, wherefrom the comparison theorem yields

$$T_{12} \leqslant T_1 \leqslant T_{11}. \tag{12}$$

The solutions to problems (9)-(11) are

$$T_{11} = x + \sum_{k=1}^{\infty} \left\{ \frac{\alpha_k}{a_1^2 \pi^2 k^2} + \left[\frac{2(-1)^{k+1}}{\pi k} - \frac{\alpha_k}{\pi^2 k^2 a_1^2} \right] \exp(-a_1^2 \pi^2 k^2 t) \right\} \sin \pi k x;$$

$$T_{12} = x + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \exp(-a_1^2 \pi^2 k^2 t) \sin \pi k x,$$

where

$$\alpha_k = \begin{cases} \dfrac{4\alpha_1}{\pi k} & k = 2n + 1, \\ 0 & k = 2n. \end{cases}$$

For the thermal fluxes at x = 0 we have, by virtue of (9) and (12), the estimate

$$q_{12} \leqslant q \leqslant q_{11}.$$

Calculations yield

$$\gamma = \max_t \frac{q_{11}}{q_{12}} = 1 + 0.5 \frac{\alpha_1}{a_1^2}.$$

The solution for T_2 is estimated analogously.

3. We will next estimate the solution to the following problem:

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(a(T) \frac{\partial T}{\partial x} \right); \tag{13}$$

$$T(0, t) = T_c; \quad T(1; t) = 1; \quad T(x, 0) = 1. \tag{14}$$

As before, we introduce a function $u = \int\limits_1^T a(y)dy$ representing the solution to the problem

$$\frac{\partial u}{\partial t} = a(T) \frac{\partial^2 u}{\partial x^2}, \tag{15}$$

$$u(0, t) = \int\limits_1^{T_c} a(y)\,dy = u_0; \quad u(1, t) = 0; \quad u(x, 0) = 0. \tag{16}$$

Let functions u_1 and u_2 satisfy conditions (16) and represent respectively the solutions to equations

$$\frac{\partial u_1}{\partial t} = a(T_c) \frac{\partial^2 u_1}{\partial x^2} = a_1 \frac{\partial^2 u_1}{\partial x^2}; \quad \frac{\partial u_2}{\partial t} = a(1) \frac{\partial^2 u_2}{\partial x^2} = a_2 \frac{\partial^2 u_2}{\partial x^2}. \tag{17}$$

Applying the Westphal theorem, we obtain

$$u_2 \leqslant u \leqslant u_1. \tag{18}$$

Solutions u_1 and u_2 are well known [10]:

$$u_i = u_0(1 - x) - \frac{2u_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin(\pi n x) \exp(-\pi^2 n^2 a_i t),$$

$$i = 1, 2.$$

For the thermal fluxes at x = 0 we have from (18) and (16)

$$q_1(0, t) \leqslant a(T) \frac{\partial T}{\partial x}\Big| \leqslant q_2(0, t).$$

The values of the quantity $\gamma = \max |q_1(0, t)/q_2(0, t)|$ at values of $\alpha = a_2/a_1 = 2.0$, 1.6, 1.4, 1.2, 1.1, and 1.05 are 1.35, 1.27, 1.19, 1.10, 1.05, and 1.02 respectively.

4. At this point it will be shown how to obtain refined estimates of solutions by successive applications of the comparison theorem.

Let it be required to estimate the solution to the following problem:

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left[(1 + 0.5T) \frac{\partial T}{\partial x} \right], \tag{19}$$

$$T(0, t) = 0; \quad T(\infty, t) = 1; \quad T(x, 0) = 1. \tag{20}$$

We introduce the function $u = (1 + 0.5T)^2$ representing the solution to the problem

$$\frac{\partial u}{\partial t} = \sqrt{u} \frac{\partial^2 u}{\partial x^2}; \tag{21}$$

$$u(0, t) = 1; \quad u(\infty, t) = 2.25; \quad u(x, 0) = 2.25. \tag{22}$$

In order to arrive at an approximation, we seek functions u_1 and u_2 which satisfy conditions (22) and equations

$$\frac{\partial u_1}{\partial t} = \frac{\partial^2 u_1}{\partial x^2}; \quad \frac{\partial u_2}{\partial t} = 1.5 \frac{\partial^2 u_2}{\partial x^2}. \tag{23}$$

According to the comparison theorem, $u_2 \le u \le u_1$. Thermal fluxes corresponding to solutions u_1 and u_2 are proportional:

at $x = 0$

$$q_1 \sim \frac{0.7}{\sqrt{t}}, \quad q_2 \sim \frac{0.57}{\sqrt{t}}. \tag{24}$$

In order to refine these estimates, it is necessary to use solutions u_1 and u_2. In order to refine the lower estimate, for example, one must use u_1. For this, one seeks the function u_3 which represents the solution to the problem

$$\frac{\partial u_3}{\partial t} = \sqrt{u_1} \frac{\partial^2 u_3}{\partial x^2};$$

$$u_3(0, t) = 1; \quad u_3(\infty, t) = 2.25; \quad u_3(x, 0) = 2.25.$$

Obviously, by virtue of (21) and the comparison theorem, the following inequalities

$$\frac{\partial u}{\partial t} \geqslant \sqrt{u_1} \frac{\partial^2 u}{\partial x^2}, \quad \frac{\partial u_2}{\partial t} \leqslant \sqrt{u_1} \frac{\partial^2 u_2}{\partial x^2},$$

hold true. For this reason, by applying the comparison theorem again, one may find that $u_2 \le u_3 \le u$. For the thermal flux at $x = 0$ we obtain the estimate:

$$q_2 \leqslant q_3 \leqslant q.$$

Numerical computations show that $q_3 \sim 0.61/\sqrt{t}$, i.e., the lower estimate has been refined. The upper estimate is refined by an analogous procedure.

5. At this point we will establish estimates for solutions to boundary-value problems in the theory of heat conduction where the maximum-value principle is not satisfied. In such a situation one can obtain a priori estimates of solutions which, however, are rather rough and unsuitable for refinement.

For instance, let the temperature distribution in an infinitely large medium with a temperature-dependent thermal conductivity and with a continuous heat source be

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(a^2 T \frac{\partial T}{\partial x}\right) + b \exp(-kx), \tag{25}$$

$$T(0, t) = T_c; \quad T(x, 0) = T_0; \quad \frac{\partial T}{\partial x}(\infty, t) = 0. \tag{26}$$

We assume that $\partial T/\partial t \le 0$ (for this we let $T_0 > T_c$). We introduce the function $u = T^2$ representing the solution to the problem:

$$\frac{\partial u}{\partial t} = a^2 \sqrt{u} \frac{\partial^2 u}{\partial x^2} + 2b\sqrt{u} \exp(-kx), \tag{27}$$

$$u(0, t) = T_c^2 = u_c; \quad u(x, 0) = T_0^2 = u_0; \quad \frac{\partial u}{\partial x}(\infty, t) = 0. \tag{28}$$

We assume that function $u(x, t)$ is bounded, denoting its as yet unknown upper and lower limits (not necessarily exact) by u_{max} and u_{min} respectively:

$$u_{\min} \leqslant u \leqslant u_{\max}. \tag{29}$$

We also introduce the functions u_1 and u_2 which respectively satisfy equations

$$\begin{aligned} \frac{\partial u_1}{\partial t} &= a^2 \sqrt{u_{\max}} \frac{\partial^2 u_1}{\partial x^2} + 2b\sqrt{u_{\min}} \exp(-kx), \\ \frac{\partial u_2}{\partial t} &= a^2 \sqrt{u_{\min}} \frac{\partial^2 u_2}{\partial x^2} + 2b\sqrt{u_{\max}} \exp(-kx) \end{aligned} \tag{30}$$

and conditions (28). Obviously, the following inequalities

$$a^2\sqrt{u_{max}}\frac{\partial^2 u}{\partial x^2}+2b\sqrt{u_{min}}\exp(-kx)\leqslant\frac{\partial u}{\partial t}$$

$$\leqslant a^2\sqrt{u_{min}}\frac{\partial^2 u}{\partial x^2}+2b\sqrt{u_{max}}\exp(-kx),$$

hold true and thus, on the basis of Westphal's comparison theory, $u_1 \leq u \leq u_2$.

An analysis of functions

$$v_1 = u_1 + \frac{2b}{a^2k^2}\sqrt{\frac{u_{min}}{u_{max}}}\exp(-kx),$$

$$v_2 = u_2 + \frac{2b}{a^2k^2}\sqrt{\frac{u_{max}}{u_{min}}}\exp(-kx), \tag{31}$$

will easily establish the validity of the following inequalities:

$$\min\left(u_0,\ u_c+\frac{2b}{a^2k^2}\sqrt{\frac{u_{min}}{u_{max}}}\right)\leqslant v_1,$$

$$u_0+\frac{2b}{a^2k^2}\sqrt{\frac{u_{max}}{u_{min}}}\geqslant v_2.$$

From (31) we obtain the following estimates for u_1 and u_2:

$$u_1\geqslant\min\left(u_0,\ u_c+\frac{2b}{a^2k^2}\sqrt{\frac{u_{min}}{u_{max}}}\right)-\frac{2b}{a^2k^2}\sqrt{\frac{u_{min}}{u_{max}}},$$

$$u_2\leqslant u_0+\frac{2b}{a^2k^2}\sqrt{\frac{u_{max}}{u_{min}}}.$$

Consequently, for the original function u we have the following estimates

$$u_0+\frac{2b}{a^2k^2}\sqrt{\frac{u_{max}}{u_{min}}}\geqslant u_2\geqslant u\geqslant u_1\geqslant\min\left[u_0,\ u_c+\frac{2b}{a^2k^2}\sqrt{\frac{u_{min}}{u_{max}}}\right]$$

$$-\frac{2b}{a^2k^2}\sqrt{\frac{u_{min}}{u_{max}}}.$$

Now u_{max} and u_{min} are determined from the system of equations

$$u_{max}=u_0+\frac{2b}{a^2k^2}\sqrt{\frac{u_{max}}{u_{min}}},$$

$$u_{min}=\min\left(u_c;\ u_0-\frac{2b}{a^2k^2}\sqrt{\frac{u_{min}}{u_{max}}}\right).$$

The exact solution to problem (30), (28) is well known [8]:

$$\frac{u_i-u_c}{u_0-u_c}=\operatorname{erfc}\left(\frac{x}{2\sqrt{m_it}}\right)-\frac{n_i}{m_ik^2(u_0-u_c)}\left[\operatorname{erfc}\left(\frac{x}{2\sqrt{m_it}}\right)\right.$$

$$-\exp(-kx)-\frac{1}{2}\exp(k^2m_it-kx)\operatorname{erfc}\left(k\sqrt{m_it}-\frac{x}{2\sqrt{m_it}}\right)$$

$$\left.+\frac{1}{2}\exp(k^2m_it+kx)\operatorname{erfc}\left(k\sqrt{m_it}+\frac{x}{2\sqrt{m_it}}\right)\right],$$

where

$$m_1=a^2\sqrt{u_{max}},\quad m_2=a^2\sqrt{u_{min}},\ n_1=2b\sqrt{u_{min}},\quad n_2=2b\sqrt{u_{max}};$$

$$\operatorname{erfc}(y)=1-\frac{2}{\sqrt{\pi}}\int_0^y\exp(-\xi^2)\,d\xi.$$

The ratio of estimates for the thermal flux at x = 0 is

$$\alpha=\frac{q_1}{q_2}$$

$$=\frac{\{-k^2m_1\Delta u-n_1[k\sqrt{\pi m_1t}(1+\exp(k^2m_1t)\operatorname{erfc}(k\sqrt{m_1t}))-2]\}\,m_2\sqrt{m_2}}{\{-k^2m_2\Delta u-n_2[k\sqrt{\pi m_2t}(1+\exp(k^2m_2t)\operatorname{erfc}(k\sqrt{m_2t}))-2]\}\,m_1\sqrt{m_1}},$$

with $\Delta u = u_0 - u_1$. For $T_C/T_0 = 2.0$, 1.8, 1.6, 1.2, 1.1, and 1.05, with $a^2 = 10^{-9}$ $m^2 \cdot sec^{-1}(°C)^{-1}$, $b = 10^{-6}$°C $\cdot sec^{-1}$, and $k = 900$ m^{-1}, the values of α are 0.5, 0.55, 0.66, 0.83, 0.91, and 0.95 respectively.

NOTATION

T is the temperature;
α^2 is the thermal diffusivity;
q is the thermal flux;
α is the heat transfer coefficient.

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